

# The Mass Quantum and Black Hole Entropy

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## Abstract

We give a method in which a quantum of mass equal to twice the Planck mass arises naturally. Then using Bose-Einstein statistics we derive an expression for the black hole entropy which physically tends to the Bekenstein-Hawking formula.

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With the introduction by Bekenstein [1] of the intriguing concept of black hole entropy and that it is proportional to the black hole area  $A$ , followed by the work of Bardeen, Carter and Hawking [2] and of Hawking [3] (see also [4, 5]), it now appears that  $\kappa/2\pi$  is the physical temperature  $T$  of a black hole,  $\kappa$  being its surface gravity. In order to make this firm, it therefore becomes imperative that a calculation of  $S$  from first principles in quantum statistics be made and that it give the Bekenstein-Hawking relation  $S = A/4$ .

In a recent publication Ashtekar *et al* [6], using a quantum theory of geometry, obtain, for a large non-rotating black hole, an expression for  $S$  which reduces to  $A/4$  with a specific choice of the Immirzi parameter [7] which enters the calculation. The *horizon* plays an important role in their method. (For other attempts at calculating the black hole entropy, see refs. [6, 8] and references cited therein.)

In the present paper we give an alternative derivation of  $S$  for a Schwarzschild black hole based on quantum statistics and obtain an expression which tends to the Bekenstein-Hawking relation. Throughout this paper we deal with only the Schwarzschild case, and use units in which  $G = c = \hbar = k = 1$ .

Another promising concept which was introduced by Bekenstein [9] is that squared irreducible mass  $M_{ir}^2$  of a black hole may be quantized in the form

$$M^2 = gn \ , \quad n = 1, 2, \dots$$

( $M_{ir} = M$  for a Schwarzschild black hole) with  $g$  a constant. The alternative form  $A = \alpha n$  is used in refs.[10, 11, 12] as  $A = 16\pi M^2$ . What struck us most about this particular quantization rule of Bekenstein was that it strongly implied the *presence* of a *simple harmonic oscillator, somewhere* in the quantum dynamics of a black hole. Below we show that the harmonic oscillator is *truly* present.

Let us start with the classical timelike geodesic equation [13]

$$\frac{1}{2} \dot{r}^2 + \frac{L^2}{2r^2} - \frac{m}{r} - \frac{mL^2}{r^3} = \frac{1}{2}(E^2 - 1) \quad (1)$$

of a test particle, where an overdot represents differentiation with respect to  $\tau$  the proper time,  $m$  is the mass that enters the Schwarzschild metric [13]

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (2)$$

$E = \left(1 - \frac{2m}{r}\right) \dot{t}$ , a conserved quantity, is interpreted as the total energy per unit rest mass of the test particle following the geodesic [13], and  $L = r^2 \dot{\phi}$ , also a conserved quantity, is the angular momentum per unit rest mass of the test particle. Equation (1) shows that it is the same [13] as that of a unit mass particle in non-relativistic mechanics [14] with kinetic energy

$$K \quad E = \frac{1}{2} \dot{r}^2 + \frac{L^2}{2r^2},$$

effective potential energy

$$V = -\frac{m}{r} - \frac{mL^2}{r^3},$$

and effective total energy

$$\bar{E} = \frac{1}{2}(E^2 - 1).$$

Hence, using the usual Schrödinger prescription [15] in (1), we obtain the corresponding stationary state quantum equation

$$\left(-\frac{1}{2}\nabla^2 - \frac{m}{r} - \frac{mL^2}{r^3}\right)\psi = \frac{1}{2}(E^2 - 1)\psi. \quad (3)$$

Eq. (3), in  $r, \theta, \phi$  coordinates, is

$$\left[-\frac{1}{2r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r}\right) + \frac{L^2}{2r^2} - \frac{m}{r} - \frac{mL^2}{r^3}\right]\psi = \frac{1}{2}(E^2 - 1)\psi. \quad (4)$$

With

$$\psi = R(r)Y_\ell^m(\theta, \phi)$$

the radial part of Eq. (4) is given by

$$\left(-\frac{1}{2r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr}\right) + \frac{\ell(\ell+1)}{2r^2} - \frac{m}{r} - \frac{m\ell(\ell+1)}{r^3}\right) R = \frac{1}{2}(E^2 - 1)R. \quad (5)$$

Note that the quantum equation (5) is characterized by  $m, E$  and  $\ell$ .

Now let us *bring in the horizon*. This is done by letting the variable  $r$  be the coordinate of the horizon and the operator  $m(r) = r/2$ .  $m$  varies as  $r$  varies. Substituting  $r/2$  for  $m$  in the last term on the LHS of Eq. (5) makes the term equal to  $\frac{-\ell(\ell+1)}{2r^2}$  which cancels the second term on the LHS,  $\frac{+\ell(\ell+1)}{2r^2}$ . Physically this means that the horizon has “swallowed” all information about the angular momentum of the test particle. In the third term  $-m/r$ , however, instead of replacing  $m$  by  $r/2$ , we replace it by its eigenvalue which is denoted by the same symbol  $m$ . (That  $m$  is indeed the eigenvalue becomes evident as one proceeds.)

And we get the equation

$$\left(-\frac{1}{2r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr}\right) - \frac{m}{r}\right) R = \frac{1}{2}(E^2 - 1)R. \quad (6)$$

To ensure that *no trace* of the test particle is left and that the variable  $r$  is the horizon coordinate, we put  $E = 0$  and obtain

$$\left(-\frac{1}{2r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr}\right) - \frac{m}{r}\right) R = -\frac{1}{2}R \quad (7)$$

as the quantum equation that is characterized by only  $m$ , the Schwarzschild mass. (It is easy to see now that if we had replaced the operator  $m$  in the term  $-m/r$  by  $r/2$  rather than its eigenvalue, there would be no  $m$  left to quantize and we would not be able to proceed any

further.) With  $m = \mu/4$  and  $U = rR(r)$ , Eq. (7) reduces to the simple form

$$\left(-\frac{1}{2} \frac{d^2}{dr^2} - \frac{\mu/4}{r}\right) U = -\frac{1}{2} U. \quad (8)$$

Eq. (8) is the quantum eigenvalue equation for the mass  $\mu$ . That  $\mu$  has the eigenvalues

$$\mu = 2(n+1)\omega \quad (9)$$

with  $\omega = 2$ ,  $n = 0, 1, 2, \dots$  is seen via the straight forward method of Mavromatis [16]:

The radial Schrödinger equation for the N-dimensional oscillator problem can be written as [17]

$$\begin{aligned} \left[ -\frac{1}{2} \left( \frac{d^2}{dr^2} - \frac{\left(\ell + \frac{N}{2} - \frac{3}{2}\right) \left(\ell + \frac{N}{2} - \frac{1}{2}\right)}{r^2} \right) + \frac{1}{2} \omega^2 r^2 \right] U_{n\ell}^{(N)}(r) \\ = \left( 2n + \ell + \frac{N}{2} \right) \omega U_{n\ell}^{(N)}(r), \end{aligned} \quad (10)$$

where  $n = 0, 1, \dots$ , and the orbital quantum number  $\ell$  is a positive interger or zero. In  $N'$  dimensions the Coulomb counterpart of Eq. (10) is

$$\left[ -\frac{1}{2} \left( \frac{d^2}{ds^2} - \frac{\left(\ell' + \frac{N'}{2} - \frac{3}{2}\right) \left(\ell' + \frac{N'}{2} - \frac{1}{2}\right)}{s^2} \right) - \frac{\alpha}{s} \right] \psi(s) = -B\psi(s), \quad (11)$$

$N'$  being the dimension of the Coulomb space,  $\ell'$  the corresponding orbital angular momentum quantum number, and  $B$  the Coulomb binding energy. Under the transformations

$$s = \rho^2, \quad \text{and} \quad \psi = \rho^{1/2} \phi, \quad (12)$$

Eq. (11) becomes

$$\left[ -\frac{1}{2} \left( \frac{d^2}{d\rho^2} - \frac{\left(2\ell' + \frac{2N'-2}{2} - \frac{3}{2}\right) \left(2\ell' + \frac{2N'-2}{2} - \frac{1}{2}\right)}{\rho^2} \right) + 4B\rho^2 \right] \phi(\rho) = 4\alpha\phi(\rho). \quad (13)$$

Comparison of Eqs. (10) and (13) shows that there is a satisfactory mapping when

$$\begin{aligned} \ell &= 2\ell' \quad , \quad N = 2N' - 2, \\ \frac{1}{2}\omega^2 &= 4B \quad , \quad (2n + \ell + \frac{N}{2})\omega = 4\alpha. \end{aligned} \tag{14}$$

Now *notice* that our Eq. (8) is Eq. (11) with

$$\ell' = 0, \quad N' = 3, \quad B = \frac{1}{2}, \quad \alpha = \mu/4; \tag{15}$$

and hence it represents a *four-dimensional harmonic oscillator* with  $\omega = 2$ , and  $\mu$  given by Eq. (9). In modern language Eq. (9) or

$$\mu_n = 2(n+1)\omega \quad , \quad n = 0, 1, 2, \dots \tag{9'}$$

says that the  $n$ -th mass ( $\mu$ ) state is occupied by  $n$  *pairs* of mass quanta, each quantum being of  $\omega = 2$  or mass equal to *twice* the Planck mass. This means that there are no mass states with odd number of quanta.

Now for calculating the entropy of a black hole of mass  $M$ , let  $2N$  mass quanta be enclosed in a volume  $V$  and their total mass be  $M = 2N\omega$ . Let us rewrite Eq. (9') as

$$\epsilon_n = \frac{\mu_n}{2} = (n+1)\omega, \quad n = 0, 1, 2, \dots \tag{16}$$

so that (i) the  $n$ th  $\epsilon$ -state contains  $n$  quanta, (ii) the thermodynamic quantity  $E$  is given by

$$E = \frac{M}{2} = N\omega, \tag{17}$$

and (iii) the thermodynamic probability of the macroscopically defined state  $(N, E)$  can simply be calculated by literally applying Bose's method as given in his original paper [18].

Let  $p_0$  be the number of vacant  $\epsilon$ -cells,  $p_1$  the number of those  $\epsilon$ -cells which contain one

quantum,  $p_2$  the number of  $\epsilon$ -cells containing two quanta, and so on. Then the probability of the state defined by the  $p_r$  is obviously

$$W = \frac{P!}{p_0!p_1!\cdots} \quad (18)$$

where

$$P = \sum_r p_r \quad (19)$$

is the total number of  $\epsilon$ -cells<sup>1</sup> over which

$$N = \sum_r r p_r \quad (20)$$

quanta are distributed. Since  $p_r$  are large numbers, we have

$$\ell n W = P \ell n P - \sum_r p_r \ell n p_r. \quad (21)$$

Then it is straightforward [18] to maximize (21) satisfying the auxiliary conditions (17) and (20), and one obtains

$$p_r = P(1 - e^{-\omega/\beta})e^{-r\omega/\beta}, \quad (22)$$

$$N = P(e^{\omega/\beta} - 1)^{-1}, \quad (23)$$

and

$$S = \frac{E}{\beta} - P \ell n(1 - e^{-\omega/\beta}). \quad (24)$$

From the condition  $\frac{\partial S}{\partial E} = \frac{1}{T}$ , one obtains  $\beta = T$ ; and (24) becomes

$$S = \frac{E}{T} - P \ell n(1 - e^{-\omega/T}). \quad (25)$$

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<sup>1</sup>The total number of  $\mu$ -cells is also  $P$ . The only difference between an  $\epsilon$ -cell and a  $\mu$ -cell is that if an  $\epsilon$ -cell contains  $r$  quanta, then the corresponding  $\mu$ -cell contains  $2r$  quanta.

For a Schwarzschild black hole for which  $T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}$  and  $A = 16\pi M^2$ , (25) becomes

$$S = \frac{A}{4} - \frac{M}{4}(e^{A/M} - 1)\ln(1 - e^{-A/M}) \quad (26)$$

which for large  $M$  physically<sup>2</sup> tends to

$$S = \frac{A}{4}. \quad (27)$$

Thus the picture of a black hole that emerges from above is:

A black hole is a Bose-Einstein ensemble of quanta of mass equal to twice the Planck mass, confined in a volume of two-sphere of radius twice the black hole mass.

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<sup>2</sup>Mathematically Eq. (26) reduces to  $S = \frac{A}{4} + \frac{M}{4}$ .



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